General Rotational Surfaces with Pointwise 1-Type Gauss Map in Pseudo- Euclidean Space $\mathbb{E}_2^4$

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Abstract

In this paper, we study general rotational surfaces in the 4- dimensional pseudo-Euclidean space $\mathbb{E}_2^4$ and obtain a characterization of flat general rotation surfaces with pointwise 1-type Gauss map in $\mathbb{E}_2^4$ and give an example of such surfaces.

Key words: Rotation surface, Gauss map, Pointwise 1-type Gauss map , pseudo-Euclidean space.

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1 Introduction

A pseudo- Riemannian submanifold M of the $m-$dimensional pseudo-Euclidean space $\mathbb{E}_s^m$ is said to be of finite type if its position vector x can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is, $x = x_0 + x_1 + ...x_k$, where x_0 is a constant map, $x_1, ..., x_k$ are non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, ..., k$. If $\lambda_1, \lambda_2,...,\lambda_k$ are all different, then M is said to be of $k-$type. This definition was similarly extended to differentiable maps in Euclidean and pseudo-Euclidean space, in particular, to Gauss maps of submanifolds [6].

If a submanifold M of a Euclidean space or pseudo-Euclidean space has 1-type Gauss map G, then G satisfies $\Delta G = \lambda (G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C. Chen and Piccinni made a general study on compact submanifolds of Euclidean spaces with finite type Gauss map and they proved that a compact hypersurface M of $\mathbb{E}^{n+1}$ has 1-type Gauss map if and only if M is a hypersphere in $\mathbb{E}^{n+1}$ [6].

Hovewer the Laplacian of the Gauss map of several surfaces and hypersurfaces such as a helicoids of the 1st,2nd and 3rdkind, conjugate Enneper's surface of the second kind in 3- dimensional Minkowski space E_1^3, generalized catenoids,

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spherical n-cones, hyperbolical n-cones and Enneper's hypersurfaces in E_1^n take the form namely,

$$\Delta G = f\,(G + C) \tag{1}$$

for some smooth function f on M and some constant vector C. A submanifold M of a pseudo-Euclidean space $\mathbb{E}_s^m$ is said to have pointwise 1-type Gauss map if its Gauss map satisfies (1) for some smooth function f on M and some constant vector C. A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector C in (1) is zero vector. Otherwise, the pointwise 1-type Gauss map is said to be of the second kind.

Surfaces in Euclidean space and in pseudo-Euclidean space with pointwise 1-type Gauss map were recently studied in [5], [7], [8], [9], [11], [12], [13], [14], [15], [17], [18]. Also Dursun and Turgay in [10] gave all general rotational surfaces in $\mathbb{E}^4$ with proper pointwise 1-type Gauss map of the first kind and classified minimal rotational surfaces with proper pointwise 1-type Gauss map of the second kind. Arslan et al. in [2] investigated rotational embedded surface with pointwise 1-type Gauss map. Arslan at el. in [3] gave necessary and sufficent conditions for Vranceanu rotation surface to have pointwise 1-type Gauss map. Yoon in [20] showed that flat Vranceanu rotation surface with pointwise 1-type Gauss map is a Clifford torus and in [19] studied rotation surfaces in the 4-dimensional Euclidean space with finite type Gauss map. Kim and Yoon in [16] obtained the complete classification theorems for the flat rotation surfaces with finite type Gauss map and pointwise 1-type Gauss map. The authors in [1] studied flat general rotational surfaces in the 4- dimensional Euclidean space $\mathbb{E}^4$ with pointwise 1-type Gauss map and they showed that a non-planar flat general rotational surfaces with pointwise 1-type Gauss map is a Lie group if and only if it is a Clifford Torus.

In this paper, we study general rotational surfaces in the 4- dimensional pseudo-Euclidean space $\mathbb{E}_2^4$ and obtain a characterization for flat general rotation surfaces with pointwise 1-type Gauss map and give an example of such surfaces.

2 Preliminaries

Let E_s^m be the $m-$dimensional pseudo-Euclidean space with signature $(s, m-s)$. Then the metric tensor g in E_s^m has the form

$$g = \sum_{i=1}^{m-s} (dx_i)^2 - \sum_{i=m-s+1}^{m} (dx_i)^2$$

where $(x_1, ..., x_m)$ is a standard rectangular coordinate system in E_s^m.

Let M be an $n-$dimensional pseudo-Riemannian submanifold of a $m-$dimensional pseudo-Euclidean space $\mathbb{E}_s^m$. We denote Levi-Civita connections of $\mathbb{E}_s^m$ and M by $\widetilde{\nabla}$ and ∇, respectively. Let e_1,...,e_n, e_{n+1},...,e_m be an adapted local orthonormal frame in $\mathbb{E}_s^m$ such that e_1,...,e_n are tangent to M and e_{n+1},...,e_m normal to

M. We use the following convention on the ranges of indices: $1 \leq i, j, k,... \leq n$, $n+1 \leq r, s, t,... \leq m$, $1 \leq A, B, C,... \leq m$.

Let ω_A be the dual-1 form of e_A defined by $\omega_A(X) = \langle e_A, X \rangle$ and $\varepsilon_A = \langle e_A, e_A \rangle = \pm 1$. Also, the connection forms ω_{AB} are defined by

$$de_A = \sum_B \varepsilon_B \omega_{AB} e_B, \quad \omega_{AB} + \omega_{BA} = 0$$

Then we have

$$\tilde{\nabla}_{e_k}^{e_i} = \sum_{j=1}^{n} \varepsilon_j \omega_{ij}(e_k) e_j + \sum_{r=n+1}^{m} \varepsilon_r h_{ik}^r e_r \tag{2}$$

and

$$\tilde{\nabla}_{e_k}^{e_s} = -\sum_{j=1}^{n} \varepsilon_j h_{kj}^s e_j + \sum_{r=n+1}^{m} \varepsilon_r \omega_{sr}(e_k) e_r, ...D_{e_k}^{e_s} = \sum_{r=n+1}^{m} \omega_{sr}(e_k) e_r, \tag{3}$$

where D is the normal connection, h_{ik}^r the coefficients of the second fundamental form h.

If we define a covariant differention $\tilde{\nabla} h$ of the second fundamental form h on the direct sum of the tangent bundle and the normal bundle $TM \oplus T^{\perp}M$ of M by

$$\left(\tilde{\nabla}_X h\right)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for any vector fields X, Y and Z tangent to M. Then we have the Codazzi equation

$$\left(\tilde{\nabla}_X h\right)(Y, Z) = \left(\tilde{\nabla}_Y h\right)(X, Z) \tag{4}$$

and the Gauss equation is given by

$$\langle R(X,Y)Z, W \rangle = \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle \tag{5}$$

where the vectors X, Y, Z and W are tangent to M and R is the curvature tensor associated with ∇. The curvature tensor R associated with ∇ is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

For any real function f on M the Laplacian Δf of f is given by

$$\Delta f = -\sum_i \left(\tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} f - \tilde{\nabla}_{\nabla_{e_i}^{e_i}} f\right) \tag{6}$$

Let us now define the Gauss map G of a submanifold M into $G(n,m)$ in $\wedge^n \mathbb{E}_s^m$, where $G(n,m)$ is the Grassmannian manifold consisting of all oriented $n-$planes through the origin of $\mathbb{E}_s^m$ and $\wedge^n \mathbb{E}_s^m$ is the vector space obtained by the exterior product of n vectors in $\mathbb{E}_s^m$. Let $e_{i_1} \wedge ... \wedge e_{i_n}$ and $f_{j_1} \wedge ... \wedge f_{j_n}$be two vectors of

$\wedge^n \mathbb{E}_s^m$, where $\{e_1, ..., e_m\}$ and $\{f_1, ..., f_m\}$ are orthonormal bases of $\mathbb{E}_s^m$. Define an indefinite inner product $\langle , \rangle$ on $\wedge^n \mathbb{E}_s^m$ by

$$\langle e_{i_1} \wedge ... \wedge e_{i_n}, f_{j_1} \wedge ... \wedge f_{j_n} \rangle = \det \left(\langle e_{i_l}, f_{j_k} \rangle \right).$$

Therefore, for some positive integer t, we may identify $\wedge^n \mathbb{E}_s^m$ with some Euclidean space $\mathbb{E}_t^N$ where $N = \begin{pmatrix} m \\ n \end{pmatrix}$. The map $G : M \to G(n, m) \subset E_k^N$ defined by $G(p) = (e_1 \wedge ... \wedge e_n)(p)$ is called the Gauss map of M, that is, a smooth map which carries a point p in M into the oriented $n-$plane in $\mathbb{E}_s^m$ obtained from parallel translation of the tangent space of M at p in $\mathbb{E}_s^m$.

3 Flat Rotation Surfaces with Pointwise 1-Type Gauss Map in E_2^4

In this section, we study the flat rotation surfaces with pointwise 1-type Gauss map in the 4-dimensional pseudo-Euclidean space E_2^4. Let M_1 and M_2 be the rotation surfaces in E_2^4 defined by

$$\varphi(t, s) = \begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & \cosh t & \sinh t & 0 \\ 0 & \sinh t & \cosh t & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix} \begin{pmatrix} 0 \\ x(s) \\ 0 \\ y(s) \end{pmatrix},$$

$$M_1 : \ \varphi(t, s) = (y(s) \sinh t, x(s) \cosh t, x(s) \sinh t, y(s) \cosh t) \tag{7}$$

and

$$\varphi(t, s) = \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix} \begin{pmatrix} x(s) \\ 0 \\ y(s) \\ 0 \end{pmatrix}$$

$$M_2 : \ \varphi(t, s) = (x(s) \cos t, x(s) \sin t, y(s) \cos t, y(s) \sin t) \tag{8}$$

where the profile curve of M_1 (resp. the profile curve of M_2) is unit speed curve, that is, $(x'(s))^2 - (y'(s))^2 = 1$. We choose a moving frame e_1, e_2, e_3, e_4 such that e_1, e_2 are tangent to M_1 and e_3, e_4 are normal to M_1 and choose a moving frame $\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4$ such that $\bar{e}_1, \bar{e}_2$ are tangent to M_2 and $\bar{e}_3, \bar{e}_4$ are normal to M_2 which are given by the following:

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{\varepsilon_1 \left(y^2(s) - x^2(s) \right)}} \left(y(s) \cosh t, x(s) \sinh t, x(s) \cosh t, y(s) \sinh t \right) \\ e_2 &= \left(y'(s) \sinh t, , x'(s) \cosh t, x'(s) \sinh t, , y'(s) \cosh t \right) \\ e_3 &= \left(x'(s) \sinh t, y'(s) \cosh t, y'(s) \sinh t, x'(s) \cosh t \right) \\ e_4 &= \frac{1}{\sqrt{\varepsilon_1 \left(y^2(s) - x^2(s) \right)}} \left(x(s) \cosh t, y(s) \sinh t, y(s) \cosh t, x(s) \sinh t \right) \end{aligned}$$

and

$$\begin{aligned}
\bar{e}_1 &= \frac{1}{\sqrt{\varepsilon_1\left(y^2(s)-x^2\left(s\right)\right)}}\left(-x\left(s\right)\sin t, x\left(s\right)\cos t, -y(s)\sin t, y(s)\cos t\right)\\
\bar{e}_2 &= \left(x'\left(s\right)\cos t, , x'\left(s\right)\sin t, y'(s)\cos t, y'(s)\sin t\right)\\
\bar{e}_3 &= \left(y'\left(s\right)\cos t, y'\left(s\right)\sin t, x'(s)\cos t, x'(s)\sin t\right)\\
\bar{e}_4 &= \frac{1}{\sqrt{\varepsilon_1\left(y^2(s)-x^2\left(s\right)\right)}}\left(y\left(s\right)\sin t, -y\left(s\right)\cos t, x(s)\sin t, -x(s)\cos t\right)
\end{aligned}$$

where $\varepsilon_1\left(y^2(s)-x^2\left(s\right)\right)>0$, $\varepsilon_1=\pm1$. Then it is easily seen that

$$\begin{aligned}
\langle e_1,e_1\rangle &= -\langle e_4,e_4\rangle=\varepsilon_1, \qquad \langle e_2,e_2\rangle=-\langle e_3,e_3\rangle=1\\
-\langle \bar{e}_1,\bar{e}_1\rangle &= \langle \bar{e}_4,\bar{e}_4\rangle=\varepsilon_1, \qquad \langle \bar{e}_2,\bar{e}_2\rangle=-\langle \bar{e}_3,\bar{e}_3\rangle=1
\end{aligned}$$

we have the dual 1-forms as:

$$\omega_1=\varepsilon_1\sqrt{\varepsilon_1\left(y^2(s)-x^2\left(s\right)\right)}dt \qquad \text{and} \qquad \omega_2=ds \tag{9}$$

and

$$\bar{\omega}_1=-\varepsilon_1\sqrt{\varepsilon_1\left(y^2(s)-x^2\left(s\right)\right)}dt \qquad \text{and} \qquad \bar{\omega}_2=ds \tag{10}$$

By a direct computation we have components of the second fundamental form and the connection forms as:

$$\begin{aligned}
h_{11}^3 &= b(s),\ h_{12}^3=0,\ h_{22}^3=c(s)\\
h_{11}^4 &= 0,\ \ h_{12}^4=b(s),\ \ h_{22}^4=0
\end{aligned} \tag{11}$$

$$\begin{aligned}
\bar{h}_{11}^3 &= -b(s),\ \bar{h}_{12}^3=0,\ \bar{h}_{22}^3=c(s)\\
\bar{h}_{11}^4 &= 0,\ \ \bar{h}_{12}^4=b(s),\ \ \bar{h}_{22}^4=0
\end{aligned} \tag{12}$$

$$\begin{aligned}
\omega_{12} &= \varepsilon_1 a(s)\omega_1,\quad \omega_{13}=\varepsilon_1 b(s)\omega_1,\quad \omega_{14}=b(s)\omega_2\\
\omega_{23} &= c(s)\omega_2,\quad \omega_{24}=\varepsilon_1 b(s)\omega_1,\quad \omega_{34}=\varepsilon_1 a(s)\omega_1
\end{aligned} \tag{13}$$

$$\begin{aligned}
\bar{\omega}_{12} &= \varepsilon_1 a(s)\bar{\omega}_1,\quad \bar{\omega}_{13}=\varepsilon_1 b(s)\bar{\omega}_1,\quad \bar{\omega}_{14}=b(s)\bar{\omega}_2\\
\bar{\omega}_{23} &= c(s)\bar{\omega}_2,\quad \bar{\omega}_{24}=-\varepsilon_1 b(s)\bar{\omega}_1,\quad \bar{\omega}_{34}=-\varepsilon_1 a(s)\bar{\omega}_1
\end{aligned} \tag{14}$$

By covariant differentiation with respect to e_1 and e_2 (resp. $\bar{e}_1$ and $\bar{e}_2$) a straightforward calculation gives:

$$\begin{aligned}
\tilde{\nabla}_{e_1}e_1 &= a(s)e_2-b(s)e_3\\
\tilde{\nabla}_{e_2}e_1 &= -\varepsilon_1 b(s)e_4\\
\tilde{\nabla}_{e_1}e_2 &= -\varepsilon_1 a(s)e_1-\varepsilon_1 b(s)e_4\\
\tilde{\nabla}_{e_2}e_2 &= -c(s)e_3\\
\tilde{\nabla}_{e_1}e_3 &= -\varepsilon_1 b(s)e_1-\varepsilon_1 a(s)e_4\\
\tilde{\nabla}_{e_2}e_3 &= -c(s)e_2\\
\tilde{\nabla}_{e_1}e_4 &= -b(s)e_2+a(s)e_3\\
\tilde{\nabla}_{e_2}e_4 &= -\varepsilon_1 b(s)e_1
\end{aligned} \tag{15}$$

and

$$
\begin{aligned}
\tilde{\nabla}_{\bar{e}_1}\bar{e}_1 &= -a(s)\bar{e}_2 + b(s)\bar{e}_3 \\
\tilde{\nabla}_{\bar{e}_2}\bar{e}_1 &= \varepsilon_1 b(s)\bar{e}_4 \\
\tilde{\nabla}_{\bar{e}_1}\bar{e}_2 &= -\varepsilon_1 a(s)\bar{e}_1 + \varepsilon_1 b(s)\bar{e}_4 \\
\tilde{\nabla}_{\bar{e}_2}\bar{e}_2 &= -c(s)\bar{e}_3 \\
\tilde{\nabla}_{\bar{e}_1}e_3 &= -\varepsilon_1 b(s)\bar{e}_1 + \varepsilon_1 a(s)\bar{e}_4 \\
\tilde{\nabla}_{\bar{e}_2}e_3 &= -c(s)\bar{e}_2 \\
\tilde{\nabla}_{\bar{e}_1}e_4 &= -b(s)\bar{e}_2 + a(s)\bar{e}_3 \\
\tilde{\nabla}_{\bar{e}_2}e_4 &= \varepsilon_1 b(s)\bar{e}_1
\end{aligned} \tag{16}
$$

where

$$a(s) = \frac{x(s)x'(s) - y(s)y'(s)}{\varepsilon_1\left(y^2(s) - x^2(s)\right)} \tag{17}$$

$$b(s) = \frac{x(s)y'(s) - x'(s)y(s)}{\varepsilon_1\left(y^2(s) - x^2(s)\right)} \tag{18}$$

$$c(s) = x''(s)y'(s) - x'(s)y''(s) \tag{19}$$

The Gaussian curvature K of M_1 and $\bar{K}$ that of M_2 are respectively given by

$$K = \varepsilon_1 b^2(s) - b(s)c(s) \tag{20}$$

and

$$\bar{K} = b(s)c(s) - \varepsilon_1 b^2(s) \tag{21}$$

If the surfaces M_1 or M_2 is flat, then (20) and (21) imply

$$b(s)c(s) - b^2(s) = 0. \tag{22}$$

Furthermore, after some computations we obtain Gauss and Codazzi equations for both surfaces M_1 and M_2

$$\varepsilon_1 a^2(s) - a'(s) = b(s)c(s) - \varepsilon_1 b^2(s) \tag{23}$$

and

$$b'(s) = 2\varepsilon_1 a(s)b(s) - a(s)c(s) \tag{24}$$

respectively.

By using (6), (15), (16) and straight-forward computation, the Laplacians ΔG and $\Delta\bar{G}$ of the Gauss map G and $\bar{G}$ can be expressed as

$$
\begin{aligned}
\Delta G = {} & -\left(3b^2(s) + c^2(s)\right)(e_1 \wedge e_2) + \left(2a(s)b(s) - \varepsilon_1 a(s)c(s) + c'(s)\right)(e_1 \wedge e_3) \\
& + \left(3a(s)b(s) - \varepsilon_1 b'(s)\right)(e_2 \wedge e_4) + 2\left(\varepsilon_1 b(s)c(s) - b^2(s)\right)(e_3 \wedge e_4)
\end{aligned} \tag{25}
$$

$$\begin{aligned}\Delta\bar{G} &= -\left(3b^2(s)+c^2(s)\right)(e_1\wedge e_2)+\left(2a(s)b(s)-\varepsilon_1 a(s)c(s)+c'(s)\right)(e_1\wedge e_3)\\ &\quad +\left(-3a(s)b(s)+\varepsilon_1 b'(s)\right)(e_2\wedge e_4)+2\left(b^2(s)-\varepsilon_1 b(s)c(s)\right)(e_3\wedge e_4)\end{aligned} \tag{26}$$

Now we investigate the flat rotation surfaces in E_2^4 with the pointwise 1-type Gauss map satisfying (1).

Suppose that the rotation surface M_1 given by the parametrization (7) is a flat rotation surface. From (20), we obtain that $b(s)=0$ or $\varepsilon_1 b(s)-c(s)=0$. We assume that $\varepsilon_1 b(s)-c(s)\neq 0$. Then $b(s)$ is equal to zero and (24) implies that $a(s)c(s)=0$. Since $\varepsilon_1 b(s)-c(s)\neq 0$, it implies that $c(s)$ is not equal to zero. Then we obtain as $a(s)=0$. In that case, by using (17) and (18) we obtain that $\alpha(s)=(0,x(s),0,y(s))$ is a constant vector. This is a contradiction. Therefore $\varepsilon_1 b(s)=c(s)$ for all s. From (14), we get

$$\varepsilon_1 a^2(s)-a'(s)=0 \tag{27}$$

whose the trivial solution and non-trivial solution

$$a(s)=0$$

and

$$a(s)=\frac{1}{-\varepsilon_1 s+c},$$

respectively. We assume that $a(s)=0$. By (24) $b=b_0$ is a constant and $c=\varepsilon_1 b_0$. In that case by using (17), (18) and (19), x and y satisfy the following differential equations

$$x^2(s)-y^2(s)=\mu \quad \mu \text{ is a constant,} \tag{28}$$

$$x(s)y'(s)-x'(s)y(s)=-\varepsilon_1 b_0\mu, \tag{29}$$

$$x''y'(s)-x'(s)y''=\varepsilon_1 b_0. \tag{30}$$

From (28) we may put

$$x(s)=\frac{1}{2}\varepsilon\left(\mu_2 e^{\theta(s)}+\mu_1 e^{-\theta(s)}\right), \quad y(s)=\frac{1}{2}\varepsilon\left(\mu_2 e^{\theta(s)}-\mu_1 e^{-\theta(s)}\right), \tag{31}$$

where $\theta(s)$ is some smooth function, $\varepsilon=\pm 1$ and $\mu=\mu_1\mu_2$. Differentiating (31) with respect to s, we have

$$x'(s)=\theta'(s)y(s), \quad y'(s)=\theta'(s)x(s) \tag{32}$$

By substituting (31) and (32) into (19), we get

$$\theta(s)=-\varepsilon_1 b_0 s+d, \quad d=const.$$

And since the curve α is a unit speed curve, we have

$$b_0^2\mu=-1.$$

Since $\mu=-\frac{1}{b_0^2}$, $y^2(s)-x^2(s)>0$. In that case we obtain that $\varepsilon_1=1$. Then we can write components of the curve α as:

$$\begin{aligned} x(s) &= \frac{1}{2}\varepsilon\left(\mu_2 e^{(-b_0 s+d)}+\mu_1 e^{-(-b_0 s+d)}\right), \\ y(s) &= \frac{1}{2}\varepsilon\left(\mu_2 e^{(-b_0 s+d)}-\mu_1 e^{-(-b_0 s+d)}\right),\ \mu_1\mu_2=-\frac{1}{b_0^2} \end{aligned} \tag{33}$$

On the other hand, by using (25) we can rewrite the Laplacian of the Gauss map G with $a(s)=0$ and $b=c=b_0$ as follows:

$$\Delta G=-4b_0^2\left(e_1\wedge e_2\right)$$

that is, the flat surface M is pointwise 1-type Gauss map with the function $f=4b_0^2$ and $C=0$. Even if it is a pointwise 1-type Gauss map of the first kind.

Now we assume that $a(s)=\frac{1}{-\varepsilon_1 s+c}$. By using $c(s)=\varepsilon_1 b(s)$ and (24) we get

$$b'(s)=\varepsilon_1 a(s)b(s) \tag{34}$$

or we can write

$$\frac{b'(s)}{b(s)}=\frac{\varepsilon_1}{-\varepsilon_1 s+c},$$

whose the solution

$$b(s)=\frac{\lambda}{|-\varepsilon_1 s+c|}, \qquad \lambda \text{ is a constant.} \tag{35}$$

By using (25) we can rewrite the Laplacian of the Gauss map G with the equations $c(s)=\varepsilon_1 b(s)$, $b'(s)=\varepsilon_1 a(s)b(s)$ and $a'(s)=\varepsilon_1 a^2(s)$

$$\Delta G=-4b^2(s)\left(e_1\wedge e_2\right)+2a(s)b(s)\left(e_1\wedge e_3\right)+2a(s)b(s)\left(e_2\wedge e_4\right). \tag{36}$$

We suppose that the flat rotational surface M_1 has pointwise 1-type Gauss map. From (1) and (36), we get

$$-4\varepsilon_1 b^2(s)=f\varepsilon_1+f\left\langle C,e_1\wedge e_2\right\rangle \tag{37}$$

$$-2\varepsilon_1 a(s)b(s)=f\left\langle C,e_1\wedge e_3\right\rangle \tag{38}$$

$$-2\varepsilon_1 a(s)b(s)=f\left\langle C,e_2\wedge e_4\right\rangle \tag{39}$$

Then, we have

$$\left\langle C,e_1\wedge e_4\right\rangle=0,\ \ \left\langle C,e_2\wedge e_3\right\rangle=0,\ \ \left\langle C,e_3\wedge e_4\right\rangle=0 \tag{40}$$

By using (38) and (39) we obtain

$$\left\langle C,e_1\wedge e_3\right\rangle=\left\langle C,e_2\wedge e_4\right\rangle \tag{41}$$

By differentiating the first equation in (41) with respect to e_1 and by using the third equation in (41) and (42), we get

$$2a(s)\langle C, e_1 \wedge e_3\rangle - b(s)\langle C, e_1 \wedge e_2\rangle = 0 \tag{42}$$

Combining (38), (39) and (42) we then have

$$f = 4\left(a^2(s) - b^2(s)\right)$$

that is, a smooth function f depends only on s. By differentiating f with respect to s and by using (35) and (27), we get

$$f' = 2\varepsilon_1 a(s) f \tag{43}$$

By differentiating (38) with respect to s and by using (15), (27), (35), (36) and (37) we have

$$a^2 b = 0$$

or from (35) we can write

$$\lambda a^3 = 0$$

Since $a(s) \neq 0$, it follows that $\lambda = 0$. Then we obtain that $b = c = 0$. Then the surface M_1 is a part of plane.

Thus we can give the following theorems.

Theorem 1. *Let M_1 be the flat rotation surface given by the parametrization (7). Then M_1 has pointwise 1-type Gauss map if and only if M is either totally geodesic or parametrized by*

$$\varphi(t,s) = \begin{pmatrix} \frac{1}{2}\varepsilon\left(\mu_2 e^{(-b_0 s+d)} - \mu_1 e^{-(-b_0 s+d)}\right)\sinh t, \\ \frac{1}{2}\varepsilon\left(\mu_2 e^{(-b_0 s+d)} + \mu_1 e^{-(-b_0 s+d)}\right)\cosh t, \\ \frac{1}{2}\varepsilon\left(\mu_2 e^{(-b_0 s+d)} + \mu_1 e^{-(-b_0 s+d)}\right)\sinh t, \\ \frac{1}{2}\varepsilon\left(\mu_2 e^{(-b_0 s+d)} - \mu_1 e^{-(-b_0 s+d)}\right)\cosh t \end{pmatrix}, \quad \mu_1\mu_2 = -\frac{1}{b_0^2} \tag{44}$$

where b_0, μ_1, μ_2 and d are real constants.

Example 1. *Let M_1 be the flat rotation surface with pointwise 1-type Gauss map given by the parametrization (44). If we take as $b_0 = -1$, $\mu_1 = -1$, $\mu_2 = 1$, $d = 0$ and $\varepsilon = 1$, then we obtain a surface as follows:*

$$\varphi(t,s) = (\cosh s \sinh t, \sinh s \cosh t, \sinh s \sinh t, \cosh s \cosh t).$$

This surface is the product of two plane hyperbolas.

Theorem 2. *Let M_2 be the flat rotation surface given by the parametrization (8). Then M_2 has pointwise 1-type Gauss map if and only if M_2 is either totally geodesic or parametrized by*

$$\varphi(t,s) = \begin{pmatrix} \frac{1}{2}\varepsilon\left(\mu_2 e^{(-b_0 s+d)} + \mu_1 e^{-(-b_0 s+d)}\right)\cos t, \\ \frac{1}{2}\varepsilon\left(\mu_2 e^{(-b_0 s+d)} + \mu_1 e^{-(-b_0 s+d)}\right)\sin t, \\ \frac{1}{2}\varepsilon\left(\mu_2 e^{(-b_0 s+d)} - \mu_1 e^{-(-b_0 s+d)}\right)\cos t, \\ \frac{1}{2}\varepsilon\left(\mu_2 e^{(-b_0 s+d)} - \mu_1 e^{-(-b_0 s+d)}\right)\sin t \end{pmatrix}, \quad \mu_1\mu_2 = -\frac{1}{b_0^2} \tag{45}$$

Example 2. *Let M_1 be the flat rotation surface with pointwise 1-type Gauss map given by the parametrization (44). If we take as $b_0 = -1$, $\mu_1 = -1$, $\mu_2 = 1$, $d = 0$ and $\varepsilon = 1$, then we obtain a surface as follows:*

$$\varphi(t, s) = \varphi(t, s) = (\cosh s \cos t, \cosh s \sin t, \cosh s \cos t, \cosh s \sin t).$$

This surface is the product of a plane circle and a plane hyperbola.

Corollary 1. *Let M be flat general rotation surface given by the parametrization (7) or (8). If M has pointwise 1-type Gauss map then the Gauss map G on M is of 1-type.*

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